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Non-linear solutions of the renormalization group equations in the large-n limit

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Abstract. The renormalization group equations of Ma for the n-vector model in the limit of large n are solved exactly. The solutions are given in terms of the non-linear scaling fields of Wegner, both in the trivial fixed-point representation and the non-trivial fixed-point representation. The effective exponent for the susceptibility crossover is computed and it is shown that the semi-microscopic theory of Riedel and Wegner is contained in the present theory as a lowest-order approximation in the most relevant scaling field.

1. Introduction

Recently considerable attention has been devoted to the study of non-linear solutions of renormalization group (RG) equations. It is well known that in the RG approach (Wilson and Kogut 1974) critical behaviour is obtained by linearizing RG equations around a fixed point. However, if one wishes to analyse more complex phenomena, such as crossover (Fisher 1974) between distinct critical behaviours, then one needs to take into account the non-linear contributions. In the RG language crossover is described in terms of fixed points of competing stability. Crossover occurs as the non-linear terms in the RG equations drive the system from the neighbourhood of the less stable fixed point towards the neighbourhood of the more stable fixed point. This mechanism has been clearly illustrated in the work of Riedel and Wegner (1974). In their paper the RG equations are modelled in order to reproduce the essential features of crossover with no reference to a microscopic analysis. Within the field theoretic RG approach in the ϕ^4 theory, the Riedel and Wegner (RW) model can be obtained (Di Castro 1975) near the critical surface as the simplest approximation to leading order in $\epsilon = 4-d$ (where d is the space dimensionality) and in the coupling.

Higher-order corrections were recently considered by Bruce and Wallace (1976). Their effective exponent for the susceptibility differs slightly from the one computed with RW equations. The same problem of Gaussian–critical crossover behaviour between the primary trivial fixed point and the secondary non-trivial Wilson–Fisher fixed point has been approached by Lawrie (1976) using a dimensional regularization procedure. Parquet diagrams and skeleton expansion have been used to study crossover from critical to classical behaviour by Natterman and Trimper (1974) and de Pasquale and Tombesi (1977) respectively. Nelson (1975) pointed out the connection of RW model with the RG recursion relations for the Landau–Wilson model. Global solutions of differential recursion relations (Wegner and Houghton 1973) within the framework of the ϵ expansion were first given by Nicoll *et al* (1975). Their solutions are built to behave properly at the infinite Gaussian fixed point besides the finite Gaussian and the non-trivial fixed point giving rise to rather complex implicit expressions.

Rudnick and Nelson (1976) instead limited their region of integration along RG flow lines up to a value of order one for the coherence distance of the block spin system. In this way quantities near criticality are related to their non-critical expressions which can be simply evaluated. Moreover the renormalized coupling remains small over the entire region of integration.

When several fixed points are considered simultaneously, special care has in fact to be devoted to maintain the solutions within the region of validity of the approximations used in deriving the equations, as for instance, in the ϵ expansion. In this case, therefore, it is also especially important to test general ideas whenever possible with exactly solvable models. Global solutions of RG equations were for instance considered by Nelson and Fisher (1975) in various one-dimensional Ising spin systems to illustrate several features of the general theory such as the non-uniqueness of both non-linear scaling field and the RG transformation.

The large-n limit (n being the number of components of the field) of the Landau–Wilson model is an important example for which one can produce exact solutions, which are relevant for the discussion of general features of crossover phenomena.

The aim of this paper, therefore, is to give global solutions of RG equations with an application to the study of crossover from the point of view of the $1/n$ expansion. We will be concerned with the spherical model limit ($n \to \infty$). The RG equations in this limit have been derived by Ma (1973). In our opinion this model is also an excellent tool for understanding the working of RG techniques in general. The fixed-point structure and the corresponding spectra of critical operators in the range of dimensionality $3 < d < 4$ are sufficiently complex to encompass non-trivial situations, while leaving the formalism manageable beyond the linear approximation.

In § 2 we present the model, and set up the formalism. The RG recursion relations of Ma (1973) and the related linear problem are considered.

In § 3 we introduce a set of non-linear scaling fields for both the non-trivial fixed point associated with spherical critical behaviour as originally discussed by Ma (1974a) and the trivial fixed point associated with Gaussian critical behaviour. This is performed by means of appropriate generating functions. The switch from one fixed point to the other is accomplished by a Legendre transformation.

In § 4 the non-linear scaling fields are used to derive the exact equations for the renormalization trajectories. The parameter space is of infinite dimensionality. The most prominent feature of the global solutions, i.e. the fixed points, the critical surface and the separatrices, which determine the behaviour of the RG trajectories, are discussed without making any approximation. To give explicit plots of these trajectories for $3 \leq d < 4$ we retain only the most relevant scaling fields. The Riedel and Wegner model emerges as the lowest-order approximation in the most relevant linear scaling field, i.e. near the critical surface. Out of the critical surface, global solutions deviate from the RW results. We then apply the inversion method of Riedel and Wegner to study the susceptibility crossover.

2. Fixed-point representations

We consider a system in d dimensions with an n-component order parameter described by the reduced Hamiltonian

$$H[\phi]=\frac{\mathcal{H}}{k_B T}=\int d^dx[(\nabla\phi)^2+U(\phi^2)]$$

where

$$(\nabla\phi)^2=\tfrac{1}{2}\sum_{i=1}^{n}(\nabla\phi_i)^2 \qquad \phi^2=\tfrac{1}{2}\sum_{i=1}^{n}\phi_i^2.$$

$U(\phi^2)$ is an O(n) invariant interaction of arbitrary order

$$U(\phi^2)=\sum_{m=1}^{\infty}\frac{u_{2m}}{m}(\phi^2)^m \tag{2.1}$$

with $u_{2m}=O(n^{1-m})$ in order to ensure the existence of the limit $n\to\infty$.

In momentum space there is a natural cut-off Λ associated with the size of the coarse graining. In this paper we consider RG transformations generated by integration over the fraction of degrees of freedom with momenta in the range $(\Lambda/s, \Lambda)$, where the group parameter s varies continuously from 1 to ∞. This yields an s-dependent effective Hamiltonian of the form

$$H'[\phi^2]=\int d^dx[(\nabla\phi)^2+U'(\phi^2)].$$

In the limit $n\to\infty$ the exact recursion relation is conveniently expressed in terms of

$$t(\phi^2)=dU/d\phi^2 \tag{2.2}$$

which up to a constant contains the same information as $U(\phi^2)$. This reads (Ma 1973)

$$t'(\phi^2)=s^2t(\rho+s^{2-d}\phi^2) \tag{2.3}$$

with

$$\rho=\frac{n}{2}K_d\int_{\Lambda/s}^{\Lambda}dk\,\frac{k^{d-1}}{k^2+t'/s^2} \tag{2.4}$$

and $K_d=2^{1-d}\pi^{-d/2}/\Gamma(d/2)$.

The fixed point solution of equation (2.3) must satisfy the functional relation

$$t^*(\phi^2)=s^2t^*(\rho^*+s^{2-d}\phi^2). \tag{2.5}$$

The critical behaviour associated with the trivial fixed point (TFP) $t_t^*(\phi^2)\equiv 0$ is of Gaussian type.

Ma (1973) has shown that for $d<4$ there is also a non-trivial fixed point (SFP) $t_c^*(\phi^2)<\infty$ which is associated with the critical behaviour of the spherical model. The function $t_c^*(\phi^2)$ will be defined in the following.

Both the previous fixed points lie on the critical surface. This is defined by the constraint on the interaction: $t(N_c)=0$, where $N_c=\frac{1}{2}nK_d\Lambda^{d-2}/(d-2)$. Out of the critical surface for large s the solution approaches the Gaussian infinite fixed point.

Once the fixed points have been located, the usual RG analysis proceeds by splitting the interaction into a fixed-point contribution and a deviation from it:

$$U=U^*+\delta U. \tag{2.6}$$

Next the perturbation is expanded in the set of eigen-operators of the linear transformation

$$\delta U = \sum_m \mu_m O_m. \tag{2.7}$$

The expansion coefficients are the linear scaling fields which under the action of the linear transformation transform as

$$\mu'_m = \mu_m s^{y_m} \tag{2.8}$$

where y_m are the corresponding scaling indices. It is clear that all the quantities $\{\mu_m, O_m, y_m\}$ are fixed-point dependent. Hence equation (2.7) leads to a fixed-point dependent representation of the interaction.

2.1. TFP representation

At the trivial fixed point $\rho_t^* = (1 - s^{2-d})N_c$. It is then obvious from equation (2.3) that the linear scaling fields are in this case obtained by expanding the interaction in powers of $(\phi^2 - N_c)$. Namely, in the TFP representation we have

$$U(\phi^2) = U(N_c) + \sum_{m=1}^{\infty} \mu_{mt}(\phi^2 - N_c)^m \tag{2.9}$$

with

$$\mu_{mt} = \frac{1}{m!} \frac{d^m U}{d(\phi^2)^m}\bigg|_{N_c} \tag{2.10}$$

$$y_{mt} = d + m(2-d) \tag{2.11}$$

i.e. $\mu_{1t} = t(N_c)$, $\mu_{2t} = \frac{1}{2}\dot{t}(N_c), \ldots$, where from now on the dots denote differentiation with respect to the argument.

2.2. SFP representation

In the neighbourhood of the SFP instead the eigen-operators are provided by the powers of $t_c^*(\phi^2)$. Hence we obtain the representation

$$U = U_c^*(\phi^2) + \sum_{m=1}^{\infty} \mu_{mc}(t_c^*(\phi^2))^m \tag{2.12}$$

with

$$y_{mc} = d - 2m. \tag{2.13}$$

The connection between the linear scaling fields $\{\mu_{mc}\}$ and the original set of coupling constants is established by evaluating at N_c the successive derivatives of equation (2.12)

$$\mu_{1c} = \frac{t(N_c)}{\dot{t}_c^*(N_c)} \tag{2.14a}$$

$$\mu_{2c} = (\dot{t}(N_c) - \dot{t}_c^*(N_c) - \ddot{t}(N_c)\mu_{1c})/2(\dot{t}_c^*(N_c))^2 \tag{2.14b}$$

$$\vdots$$

At the TFP $(\mu_{mt} = 0, \forall m)$ the μ_{mc} take the values $\mu_{1c}^{*t} = 0$, $\mu_{2c}^{*t} = -1/2\dot{t}_c^*(N_c), \ldots$. In the TFP representation the SFP ($\mu_{mc}=0, \forall m$) is specified by the coordinates $\mu_{1t}^{*c}=0$, $\mu_{2t}^{*c}=\frac{1}{2}i(N_c), \ldots$.

It is also easy to connect the two representations by giving the μ_t as functions of the μ_c.

3. Flow in interaction space

The representations (2.9), (2.12) although obtained from a linear analysis, hold irrespective of the distance from a fixed point. Outside the linear region, however, the non-linear character of the RG transformation changes the s dependence of the expansion coefficients from the pure power law (2.8) to a much more complicated behaviour. Thus in order to construct the functions $\{\mu_{mt}(s)\}$, $\{\mu_{mc}(s)\}$ one must solve the full non-linear problem related to the RG transformation. At this point it is very useful to introduce the concept, due to Wegner (1972), of non-linear scaling fields.

These are parameters $\{g_m\}$ which transform like

$$g'_m = g_m s^{y_m} \tag{3.1}$$

exactly. One expects that there exists a set of non-linear scaling fields for every fixed point and that in the neighbourhood of each fixed point these coincide with the corresponding linear scaling fields. More explicitly in our case one expects to find a set $\{g_{mc}\}$ of non-linear scaling fields for the SFP representation and a set $\{g_{mt}\}$ for the TFP representation which satisfy, in the neighbourhood of the fixed points, the initial conditions

$$g_{mc} = \mu_{mc} + O(\mu^2) \tag{3.2a}$$

$$g_{mt} = \mu_{mt} + O(\mu^2). \tag{3.2b}$$

The parametric equations $\{\mu_m(s)\}$ of the renormalization trajectories are obtained once the connection between linear scaling fields and non-linear scaling fields is known in any one of the possible representations

$$\mu_m = \mu_m\{g_i\}. \tag{3.3}$$

In fact from equation (3.1) one trivially obtains

$$\mu_m(s) = \mu_m\{g_i s^{y_i}\}. \tag{3.4}$$

General procedures for the construction of the non-linear scaling fields are not known and the construction of the functions (3.3) in closed form is a difficult problem (Wegner 1972, Ma 1974a).

Nevertheless in the large-n limit it turns out to be possible to produce generating functions for the non-linear scaling fields. We consider the function

$$G(t) = U - \phi^2 t + \tfrac{1}{2}nK_d \int_0^\Lambda dk\, k^{d-1} \ln(k^2+t) + Ct^{d/2} \tag{3.5}$$

with

$$C = \frac{n}{d}K_d \int_0^\infty dx \frac{x^{d-3}}{1+x^2}. \tag{3.6}$$

Using a result of Ma (1974a) one can show that this transforms according to

$$G'(t') = s^d G(t'/s^2). \tag{3.7}$$

Therefore the non-linear scaling fields of the SFP representation are obtained as coefficients in the power expansion

$$G(t) = \sum_m g_{mc} t^m. \tag{3.8a}$$

The generating function for the set $\{g_{mt}\}$ in turn is constructed from $G(t)$ through a Legendre transformation. Define†

$$\tau(\phi^2) = -\frac{\partial G}{\partial t} = \phi^2 - N(t) = -\sum_m m g_{mc} t^{m-1} \tag{3.8b}$$

where

$$N(t) = \frac{n}{2} K_d \int_0^\Lambda \mathrm{d}k \frac{k^{d-1}}{k^2+t} + \frac{d}{2} C t^{(d/2)-1} \xrightarrow[t\to 0]{} N_c. \tag{3.9}$$

One has that the function

$$\Gamma(\tau) = G + \tau t \tag{3.10}$$

transforms according to

$$\Gamma'(\tau') = s^d \Gamma(s^{2-d}\tau'). \tag{3.11}$$

Hence the non-linear scaling fields are again obtained as coefficients in a power expansion

$$\Gamma(\tau) = \sum_m g_{mt} \tau^m \tag{3.12a}$$

with

$$\frac{\partial \Gamma}{\partial \tau} = t(\phi^2) = \sum_m m g_{mt} \tau^{m-1}. \tag{3.12b}$$

In the following we shall mainly work in the TFP representation. In this representation the construction of the basic set of equations (3.3) is much simpler. $t(N_c), \dot{t}(N_c), \ldots$ can in fact be easily expressed in terms of the g_{mt} by evaluating equation (3.12*b*) and its derivatives with respect to ϕ^2 at $\phi^2 = N_c$. Thus the first two equations read:

$$\mu_{1t} = \sum_{m=1}^{\infty} m g_{mt} (\tau(\mu_{1t}))^{m-1} \tag{3.13a}$$

$$\mu_{2t} = \frac{1}{2} \frac{\sum_{m=2}^{\infty} m(m-1) g_{mt} (\tau(\mu_{1t}))^{m-2}}{1 + \sum_{m=2}^{\infty} m(m-1) g_{mt} (\tau(\mu_{1t}))^{m-2} \dot{N}(\mu_{1t})} \tag{3.13b}$$

where

$$\tau(\mu_{1t}) = N_c - N(\mu_{1t}), \qquad \tau(0) = 0.$$

† At the SFP all $g_{mc} = 0$. Thus equation (3.8*b*) reduces to $\phi^2 = N(t_c^*)$ which defines the inverse function of $t_c^*(\phi^2)$ and coincides with equation (4.22) of Ma (1973).

After some algebra

$$\tau(\mu_{1t}) = -N_c(d-2)\left(\frac{\mu_{1t}}{\Lambda^2}\right)^{(d/2)-1} \int_{1/\sqrt{(\mu_{1t}/\Lambda^2)}}^{\infty} dx \frac{x^{d-3}}{x^2+1}. \tag{3.14}$$

Equations (3.13) and the similar equations for the other scaling fields give the exact solution of the RG equations (3.3), (3.4), whose content can now be analysed over the entire parameter space $\{\mu_m\}$ and not only near the relevant fixed point.

In the next section we shall discuss their most significant features.

In the SFP representation the equations corresponding to (3.13) are obtained by evaluating equation (3.8*b*) and its derivatives at $\phi^2 = N_c$:

$$\sum_m m g_{mc}\mu_{1t}^{m-1} = -\tau(\mu_{1t}) \tag{3.15a}$$

$$2\sum_m m(m-1)g_{mc}\mu_{1t}^{m-2}\mu_{2t} = -1 + 2\dot{N}(\mu_{1t})\mu_{2t} \tag{3.15b}$$

$$\vdots$$

These equations can be used to switch to the SFP representation whenever it is required.

4. Trajectories and crossover

In order to gain some insight into the solution let us consider a few limiting cases. The fixed points, the critical surface and the separatrices, namely all the elements which determine the structure of the flow, are characterized by special values of the non-linear scaling fields. Thus the coordinates of the TFP ($\mu_{1t}^{*t} = \mu_{2t}^{*t} = \cdots = 0$) and of the SFP ($\mu_{1t}^{*c} = 0$, $\mu_{2t}^{*c} = \frac{1}{2}t_c^*(N_c) = 1/2\dot{N}(0), \ldots$) are obtained from equations (3.13) by setting respectively $g_{mt} = 0$ for all m or $g_{1t} = 0$, $g_{2t} = \infty$ (i.e. all $g_{mc} = 0$ in equation (3.15)). Similarly the condition for criticality, $\mu_{1t} = t(N_c) = 0$ (or $\mu_{1c} = 0$ in the SFP representation), is satisfied in equation (3.13*a*) setting $g_{1t} = 0$ with all other g_{mt} arbitrary. Since τ vanishes at $\mu_{1t} = 0$, equation (3.13*b*) reduces to

$$\mu_{2t} = \frac{g_{2t}}{1 + g_{2t}/\mu_{2t}^{*c}}. \tag{4.1}$$

For $d < 4$, $\mu_{2t}(s) \to \mu_{2t}^{*c}$ as $s \to \infty$, i.e. the SFP is more stable than the TFP.

Equation (4.1) also describes a changeover from a power law behaviour of Gaussian type $\mu_{2t}(s) \sim s^{y_{2t}} g_{2t}$ in the neighbourhood of the TFP ($g_{2t} = 0$), to the power law behaviour of the spherical type $\mu_{2c}(s) \sim s^{y_{2c}} g_{2c}$ in the neighbourhood of the SFP ($g_{2t} = \infty$). This is easily checked. In fact on the critical surface $\mu_{1t} = \mu_{1c} = 0$, μ_{2c} and g_{2c} are related to μ_{2t} and g_{2t} by the simple relations:

$$\mu_{2c} = \frac{\mu_{2t} - \mu_{2t}^{*c}}{4\mu_{2t}^{*c}}, \qquad g_{2c} = \frac{1}{4g_{2t}}$$

which follow from equation (2.14*b*) and equations (3.13*b*), (3.15*b*) respectively.

The next interesting topological objects are the separatrices. The Gaussian separatrix is characterized by all $g_{mt} = 0$ except g_{1t}. Hence from equations (3.13)

$$\mu_{mt} = g_{1t}\delta_{m1} \tag{4.2}$$

namely $\mu_{1t}(s)$ scales exactly as $s^{y_{1t}}$ while all other μ_{mt} vanish identically.

More interesting is the spherical separatrix, namely the trajectory issuing from the SFP. This is characterized by all $g_{mc}=0$ except g_{1c}. From equations (3.15) it follows that the behaviour of μ_{1t} and μ_{2t} along the separatrix is governed by

$$g_{1c}+\tau(\mu_{1t})=0 \tag{4.3a}$$

$$\mu_{2t}=1/2\dot{N}(\mu_{1t}). \tag{4.3b}$$

In the neighbourhood of the SFP, $\tau(\mu_{1t})$ is linear in μ_{1t} as it follows from equation (3.14)

$$\tau(\mu_{1t})\sim\frac{d-2}{d-4}\frac{N_c}{\Lambda^2}\mu_{1t}=-\frac{\mu_{1t}}{2\mu_{2t}^{*c}}. \tag{4.4}$$

This gives $\mu_{1t}(s)=2\mu_{2t}^{*c}\mu_{1c}(s)\simeq 2\mu_{2t}^{*c}g_{1c}s^{y_{1c}}$, however the non-linear terms in $\tau(\mu_{1t})$ change this behaviour, until in the asymptotic region $g_{1c}\gg 1$, $\tau(\mu_{1t})\sim\mu_{1t}^{(d/2)-1}$ and the changeover to the Gaussian power law $s^{y_{1t}}$ occurs. Similarly equation (4.3*b*) shows that $\mu_{2t}\to\mu_{2t}^{*c}$ as $\mu_{1t}\to 0$, whereas it switches to the Gaussian power law behaviour $\mu_{2t}(s)\sim s^{y_{2t}}$ as $\mu_{1t}\to\infty$.

In general in equations (3.13) appear all the non-linear scaling fields g_{mt}. In the range of dimensionality $3<d<4$ the relevant fields are g_{1t}, g_{2t}. The simplest approximation containing crossover is obtained by keeping these two fields and discarding all others. The resulting equations are

$$\mu_{1t}=g_{1t}+2\tau(\mu_{1t})g_{2t} \tag{4.5a}$$

$$\mu_{2t}=\frac{g_{2t}}{1+2\dot{N}(\mu_{1t})g_{2t}}. \tag{4.5b}$$

If $\tau(\mu_{1t})$ was linear in μ_{1t} these equations would be analogous to the RW model equations with special values of the critical indices y_{1t}, y_{2t}, y_{1c}, y_{2c} appropriate to the trivial and the spherical fixed point. On the other hand, $\tau(\mu_{1t})$ is a very involved function of μ_{1t} and the dimensionality d, which, as already stressed in equation (4.4), vanishes linearly with μ_{1t}. Hence the Riedel and Wegner (1974) theory is here contained as a lowest-order approximation in μ_{1t}, i.e. in the neighbourhood of the critical surface, where our equations reduce to the RW equations:

$$\mu_{1t}=\frac{g_{1t}}{1+g_{2t}/\mu_{2t}^{*c}},\qquad \mu_{2t}=\frac{g_{2t}}{1+g_{2t}/\mu_{2t}^{*c}}. \tag{4.6}$$

It can be shown easily† that the same result can be obtained retaining the dominant term in $\epsilon=4-d$. The flow generated by equations (4.5) for $d=3$‡ and $d=3\cdot5$ is displayed in figures 1 and 2 and compared with the RW flow generated by equations (4.6). Away from the critical surface the results coincide only on the separatrix emerging from the TFP, where μ_{1t} scale exactly in both cases. The deviations become larger as we move towards the spherical separatrix where the RW results coincide with the present one only at the SFP. As we have already stressed, μ_{1t} and μ_{2t} do not simply scale on this separatrix and a changeover takes place from spherical to Gaussian behaviour. In the RW case instead (vertical line, at $\mu_2=1$ in figures 1 and 2) μ_{2t} is identically equal to μ_{2t}^{*c} and μ_1 scales exactly as $s^{y_{1c}}$. A comparison between figures 1 and 2 shows that, according to the previous analysis, differences are smaller as d increases towards $d=4$.

† To this purpose it is useful to express $\tau(\mu_{1t})$ in terms of the function Φ defined by the equation (5.24) of Ma (1974b).

‡ At $d=3$ the effect of the marginal field g_{3t} is disregarded for simplicity.

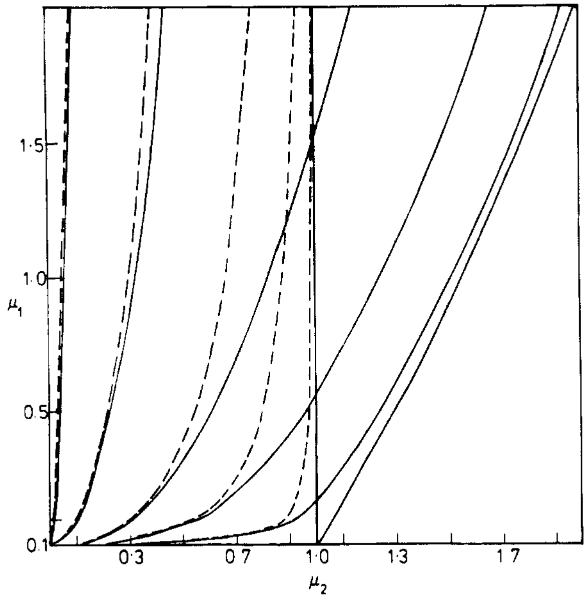

Figure 1. Flow diagram for $d = 3$. The full curves represent the renormalization trajectories in the full non-linear calculation. The broken curves are the renormalization trajectories in the RW approximation. The separatrices are the lines emanating from the point $\mu_2 = 1$. $\mu_1 = \mu_{1t}/\Lambda^2$, $\mu_2 = \mu_{2t}/\mu_{2t}^{*c}$.

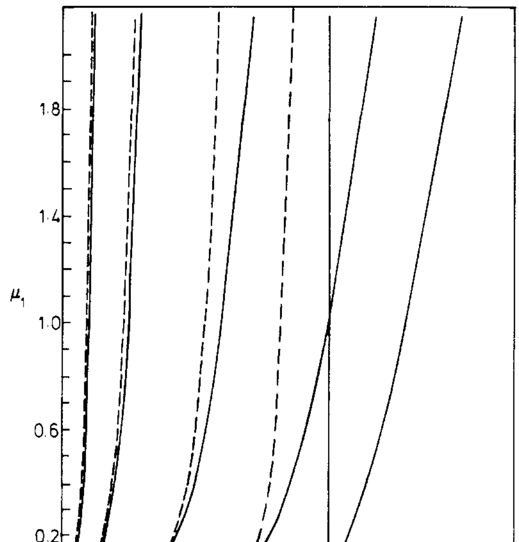

Figure 2. Flow diagram for $d = 3{\cdot}5$. The full curves are the renormalization trajectories in the full non-linear calculation. The broken curves represent the renormalization trajectories in the RW approximation. The separatrices are the lines emanating from the point $\mu_2 = 1$. $\mu_1 = \mu_{1t}/\Lambda^2$, $\mu_2 = \mu_{2t}/\mu_{2t}^{*c}$.

In figure 3 the same flow lines for $d = 3$ according to equations (4.5) are displayed in terms of reduced variables, which allow for a more compact graphical representation and make clear the globality of the solution here considered.

Once the flow trajectories are known the crossover effects can be evaluated over the entire $\{\mu(s)\}$ space. As an illustration we display in figure 4 the susceptibility crossover

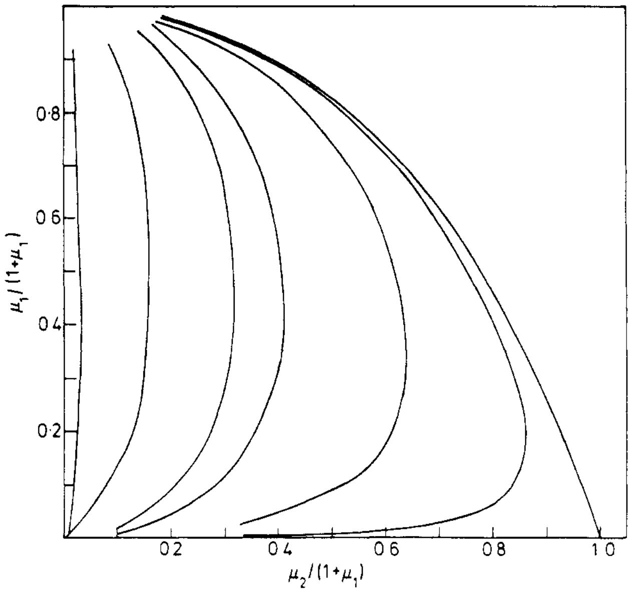

Figure 3. Flow diagram for $d = 3$ in the reduced coordinates reference frame.

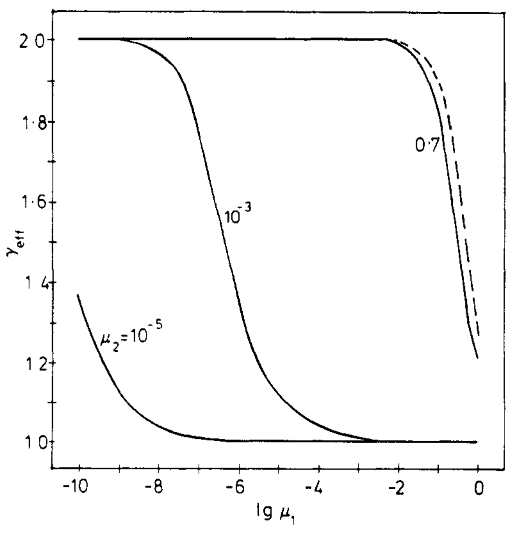

Figure 4. The effective exponent γ_{eff} for the Gaussian–spherical crossover. The full curves are the results of the full non-linear calculation. The broken curve represents γ_{eff} in the RW approximation.

along the μ_{2t} = constant paths. The effective exponent

$$\gamma_{\text{eff}} = -\frac{\partial \ln \chi(\mu_{1t}, \mu_{2t})}{\partial \ln \mu_{1t}}\bigg|_{\mu_{2t}} \tag{4.7}$$

is computed extending the Riedel and Wegner (1974) inversion method to the present case. For convenience this is presented in the appendix. As expected the effective exponent computed from the full non-linear solution behaves in the same way as the exponent computed in the RW approximation for μ_1, μ_2 small. On the other hand for $\mu_2 \sim \mu_2^{*c}$ and $\mu_1 \sim O(1)$ there is a small quantitative difference in the crossover region due to the different structure of the flow diagram.

Appendix

Starting from the generalized scaling relation

$$\chi(\mu_1, \mu_2) = s^2 \chi(\mu_1(s), \mu_2(s)) \tag{A.1}$$

where $\mu_1 = \mu_{1t}/\Lambda^2$ and $\mu_2 = \mu_{2t}/\mu_{2t}^{*c}$, define $\hat{s}$ and $\hat{\mu}_2$ by

$$\mu_1(\hat{s}) = 1 \tag{A.2}$$

$$\mu_2(\hat{s}) = \hat{\mu}_2 \tag{A.3}$$

and assume that the line $\mu_1 = 1$ lies outside the critical region. Then setting $s = \hat{s}$ in equation (A.1):

$$\chi(\mu_1, \mu_2) = \hat{s}^2 \bar{\chi}(\hat{\mu}_2) \tag{A.4}$$

where $\bar{\chi}(\hat{\mu}_2) = \chi(1, \hat{\mu}_2)$ is the non-critical susceptibility. This can be computed by standard methods, while the critical behaviour has been segregated in $\hat{s}$. Computing $\hat{s}$ from equation (4.5*a*) we obtain

$$\chi(\mu_1, \mu_2) = g_1^{-\gamma_t} (f(\phi_t, c))^{2/y_{2t}} \bar{\chi}(\hat{\mu}_2) \tag{A.5}$$

with

$$\gamma_t = 1 \tag{A.6}$$

$$c = -a(g_2/g_1^{\phi_t}) \tag{A.7}$$

$$a = 2\tau(1), \qquad \phi_t = y_{2t}/y_{1t} \tag{A.8}$$

and f is the solution of

$$f = (1 + cf)^{\phi_t}. \tag{A.9}$$

The dependence of g_1 and c on μ_1, μ_2 is obtained inverting equations (4.5*a*), (4.5*b*). Note that in terms of g_1, g_2 equation (A.5) is in the scaling homogeneous form. This form is useful in order to analyse tricritical scaling and tricritical crossover.

In the region around the TFP the renormalization group invariant c takes all values from 0 to ∞ and it is precisely for $c \to \infty$ that the scaling function produces crossover from tricritical to ordinary critical behaviour.

First notice that sufficiently close to the TFP $g_1 \sim \mu_1$, $g_2 \sim \mu_2$. Hence equation (A.5) can be rewritten in the form

$$\chi(\mu_1, \mu_2) = \mu_1^{-\gamma_t} (f(\phi_t, c))^{2/y_{2t}} \bar{\chi}(\hat{\mu}_2) \tag{A.10}$$

with

$$c=-a(\mu_2/\mu_1^{\phi_t}). \tag{A.11}$$

Next, from equation (A.9) one has

$$f(\phi_t, c)=\begin{cases}1+\phi_t c, & c \ll 1\\ c^{t/(1-\phi_t)}, & c \gg 1.\end{cases} \tag{A.12}$$

Hence, in the asymptotic tricritical region ($c \ll 1$)

$$\chi(\mu_1, \mu_2) \sim \mu_1^{-\gamma_t} \tag{A.13}$$

while in the asymptotic critical region ($c \gg 1$) one has the double scaling form

$$\chi(\mu_1, \mu_2) \sim \mu_1^{-\gamma_c} \mu_2^{(\gamma_c-\gamma_t)/\phi_t} \tag{A.14}$$

with

$$\gamma_c = \frac{2}{d-2}. \tag{A.15}$$

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